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The problem considered is the characterization of the quasi-injective abelian group as a direct sum of known groups, using only group theoretical methods.

Let $Z(n)$ denote the group of integers modulo n and let $Z(p^\infty) = \bigcup_{i=1}^{\infty} Z(p^i)$ for p a prime number. It was found that if G is a quasi-injective abelian group then either G is injective, in which case it has been characterized in the literature, or else there exists an index set I and sets A_i, $i \in I$, such that $\{p_i\}_{i \in I}$ is a set of distinct primes, n_i is a positive integer or $n_i = \infty$ for each $i \in I$, and G is isomorphic to $\oplus \sum_{i \in I} G_i$ where $G_i = \oplus \sum_{A_i} Z(p_i^{n_i})$ for each $i \in I$.

ON QUASI-INJECTIVE ABELIAN GROUPS

by

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CHAPTER I
INTRODUCTION

It is a well-known theorem that any abelian group G satisfying G = nG for every positive integer n is a direct summand of every abelian group H which contains G as a subgroup. Baer (2) generalized this concept to what he called "abelian groups admitting a ring of operators." Later his work was put in the context and language of module theory with the resulting theorem that a left R-module M is a direct summand of every left R-module N containing M if and only if M has the property that if $0 \longrightarrow P \xrightarrow{f} Q$ is any exact sequence of left R-modules and if g is an R-homomorphism from P into M then there exists an R-homomorphism h from Q into M such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "f"] \arrow[d, "g"'] & Q \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

commutes, in the sense that hf = g (3, p.389). A module having this latter property is said to be an *injective module*. Generalizing a theorem of Baer, Eckmann and Schopf proved that every R-module M can be embedded in a uniquely determined smallest injective module, called the <u>injective</u> <u>envelope</u> of M, and discovered some important properties of this module (3, p.390).

Arising from the definition of injective module is the concept of a quasi-injective module. A left R-module M is said to be <u>quasi-injective</u> provided if $0 \longrightarrow P \xrightarrow{f} M$ is an exact sequence of left R-modules and g is an R-homomorphism from P into M then there exists an R-homomorphism h from M into itself such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

commutes. Quasi-injective modules have been studied by Johnson and Wong (6), Faith and Utumi (4), and Harada (5).

An abelian group can be thought of as a module over the ring of integers. Thus we can talk about injective and quasi-injective abelian groups. It can be shown that an abelian group G is injective if and only if it is divisible, that is, if and only if $G = nG$ for every positive integer n. Divisible abelian groups have been completely characterized using the methods of group theory (7, p.10). Thus all injective abelian groups are known.

It is the purpose of this thesis to arrive at a characterization of the quasi-injective abelian groups using the methods of group theory. The main tool will be Theorem 3.1, a result of Johnson and Wong (6), which says that an abelian group G is quasi-injective if and only if every homomorphism from the injective envelope of G into itself maps G onto a subgroup of G. In order to apply this theorem we must be familiar with divisible, or injective, abelian groups, and injective envelopes in particular. These are treated in Chapter II. Using the results of Chapter II and Theorem 3.1, we characterize quasi-injective torsion abelian groups in Chapter III and we prove that a quasi-injective torsion-free abelian group must be injective. We then show that any quasi-injective abelian group that is not injective must be a torsion group. Since all injective abelian groups have been completely characterized, this gives us a complete characterization of quasi-injective abelian groups.

CHAPTER II

INJECTIVE ABELIAN GROUPS

Section 1: The Equivalence of Injective and Divisible Groups and Important Examples

To avoid repetition, we shall throughout this paper use the term "group" to mean "abelian group". For a development of the basic properties of abelian groups see (8).

2.1.1 DEFINITION. The sequence of groups and of group homomorphisms

$$\cdots \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} G_{i+2} \xrightarrow{f_{i+2}} \cdots$$

is said to be exact if the image of f_i equals the kernel of f_{i+1} for every i.

Note that it follows from the definition that the sequence of groups $0 \longrightarrow L \xrightarrow{f} M$ is exact if and only if f is one-to-one, since there is only one homomorphism possible from 0, the trivial group, to L.

2.1.2 DEFINITION. An abelian group G is said to be *injective* provided if $0 \longrightarrow L \xrightarrow{f} M$ is an exact sequence of abelian groups and if g is a group homomorphism from L into G, then there exists a group homomorphism h from M into G such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, "h"] \\ & G & \end{tikzcd}$$

commutes, in the sense that $hf = g$.

Since f, in the exact row $0 \longrightarrow L \xrightarrow{f} M$, is one-to-one, L is isomorphic to a subgroup of M, namely $f(L)$. Thus we may assume in the above definition that L is a subgroup of M and that $f = i$, the natural inclusion map. In this case $hi = g$ and we say that h is an *extension* of g to M.

2.1.3 DEFINITION. A group G is said to be divisible if $nG = G$ for every nonzero integer n. In other words, if $x \in G$ and n is any nonzero integer then there exists a $y \in G$ such that $ny = x$.

We will now show that a group is injective if and only if it is divisible. For this and several other proofs that follow we will use Zorn's Lemma in the following form.

2.1.4 ZORN'S LEMMA. Let P be a nonempty partially ordered set with the property that every simply ordered subset, or chain, in P has an upper bound in P. Then P contains a maximal element.

A proof of Zorn's Lemma and its equivalence to the Axiom of Choice can be found in (1, Chapter 4). The proof of the following theorem has the same general outline as the proof found in (3, p.387).

2.1.5 THEOREM. A group G is injective if and only if it is divisible.

Proof: Let G be a divisible group, M a group with subgroup L, and g a group homomorphism from L into G. Let S denote the set of all extensions of g to subgroups of M containing L. Since we can consider g as an extension of itself to L, S is not empty. Since every homomorphism is a set of ordered pairs, S is partially ordered by set inclusion. Note that for $g_1, g_2 \in S$, $g_1 \subseteq g_2$ if and only if the domain of g_1 is a subset of the domain of g_2 and g_2 agrees with g_1 on the domain of g_1. Let $\{g_i\}_I$ be a chain in S. Let D_i denote the domain of g_i for each $i \in I$. Define a function g^* with domain $D = \bigcup_{i \in I} D_i$ as follows. Let $x \in D$. Choose $i \in I$ such that $x \in D_i$, and define $g^*(x) = g_i(x)$. To see that g^* is a well-defined function, suppose $x \in D_i \cap D_j$ for $i \neq j$. We may assume $i < j$. Then $g_i(x) = g_j(x)$ since g_j agrees with g_i on D_i. Thus there is no ambiguity in the definition of g^*. Since $g_i \subseteq g^*$ for every $i \in I$, g^* is an upper bound of the chain $\{g_i\}_I$. By Zorn's Lemma, since every chain in S has an upper bound, S has a maximal element. Let h denote the maximal element of S and H the domain of h. We need only prove that $H = M$. We shall do this by showing that if H is a proper subgroup of M we contradict the maximality of h. Suppose H is a proper subgroup of M. Let $x \in M$ such that $x \notin H$. Let Z denote the group of integers and let $Zx = \{ax \mid a \in Z\}$. Then $Zx + H$ is a subgroup of M containing L, since H contains L. If $Zx + H$ is a direct sum, i.e., if $Zx \cap H = 0$, then define a homomorphism h^* from $Zx \oplus H$ into G by $h^*(ax + y) = h(y)$ for all $a \in Z$ and $y \in H$. Now $h^* \in S$ since h^* agrees with h on H and h agrees with g on L. But, since $x \notin H$, $h \subsetneq h^*$, contradicting the maximality of h.

Now suppose $Zx \cap H \neq 0$. Let m be the smallest positive integer such that $mx \in Zx \cap H$. Then $h(mx) \in G$. Since G is divisible there exists $z \in G$ such that $h(mx) = mz$. Define a mapping h^* from $Zx + H$ into G by $h^*(ax + y) = az + h(y)$ for all $a \in Z$, $y \in H$. We must show that h^* is a well-defined function. If $ax + y = 0$ then $ax + y - y = ax \in H$ since $y \in H$. We can write $a = pm + r$ for some integers p and r, with $0 \leq r < m$. But $ax \in H$ and $mx \in H$, so that $ax - pmx = (a - pm)x = rx \in H$. Thus $r = 0$ by choice of m, and $a = pm$. We have $ax + y = pmx + y$ with both summands in H, and $h^*(pmx + y) = pmz + h(y) = ph(mx) + h(y) = h(pmx + y) = h(0) = 0$. Thus h^* is a well-defined function. Clearly h^* is a homomorphism, so that $h^* \in S$ and $h \subsetneq h^*$, again contradicting the maximality of h. Thus we must have $H = M$, proving that G is injective.

Suppose now that G is injective. Let n be a nonzero integer and $x \in G$. Define a homomorphism from Zn into G by $g(mn) = mx$ for all $m \in Z$. Then since G is injective there exists a homomorphism h from Z into G such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & Zn \arrow[r, "i"] \arrow[d, "g"'] & Z \arrow[dl, "h"] \\ & G & \end{tikzcd}$$

commutes, where i is the natural inclusion map. Let $y = h(1)$. Then $ny = nh(1) = h(n) = hi(n) = g(n) = x$. Thus $nG = G$ and G is divisible.

We now look at two important examples of divisible, or injective, groups.

<u>2.1.6 PROPOSITION</u>. Let Q denote the additive group of rational numbers. Then Q is divisible.

Proof: Let $x \in Q$ and n a nonzero integer. Then $\frac{x}{n} \in Q$ and $n(\frac{x}{n}) = x$, so that Q is divisible.

<u>2.1.7 DEFINITION</u>. For any positive integer n define $Z(n) = \{\frac{x}{n} + Z \mid x \text{ is any integer}\}$. We call Z(n) the <u>group of integers modulo n</u>. Let p be a fixed prime and consider the infinite chain of groups

$$Z(p) \subsetneq Z(p^2) \subsetneq \ldots \subsetneq Z(p^n) \subsetneq \ldots$$

Define $Z(p^{\infty}) = \bigcup_{n=1}^{\infty} Z(p^n)$.

<u>2.1.8 PROPOSITION</u>. The group $Z(p^{\infty})$ is a divisible group for any prime p.

Proof: It is easily seen that $Z(p^{\infty})$ is an abelian group since it is the union of a chain of abelian groups.

Let $X \in Z(p^{\infty})$ and let n be a nonzero integer. We want to show that there exists $Y \in Z(p^{\infty})$ such that $nY = X$.

First suppose that n is relatively prime to p. Since $X \in \bigcup_{n=1}^{\infty} Z(p^n)$, there exists a positive integer k such that $X \in Z(p^k)$. Thus the order of X must divide p^k, the order of $Z(p^k)$, and so is equal to p^m for some integer $1 \leq m \leq k$. Since n is relatively prime to p it is also relatively prime to p^m. Thus there exist integers a and b such that $an + bp^m = 1$, and we have $X = (an + bp^m)X = anX + bp^mX = anX = n(aX)$. So there exists $Y \in Z(p^{\infty})$, namely $Y = aX$, such that $X = nY$.

Now suppose n is not relatively prime to p. Then we can write $n = mp^t$ for some positive integer t and integer m relatively prime to p. As shown above, there exists $Y \in Z(p^{\infty})$ such that $mY = X$. Now $Y \in Z(p^s)$ for some positive integer s, i.e., $Y = \frac{x}{p^s} + Z$ for some integer x. Let $Y' = \frac{x}{p^{t+s}} + Z$. Then $nY' = mp^tY' = mY = X$. Thus for any nonzero integer n, $nZ(p^{\infty}) = Z(p^{\infty})$, and so $Z(p^{\infty})$ is divisible.

Section 2: Some Important Results on Divisible, or Injective, Groups

2.2.1 PROPOSITION. The homomorphic image of a divisible group is divisible.

Proof: Let G be a divisible group and f a homomorphism from G onto the group H. Let n be a nonzero integer and $x \in H$. Then $x = f(y)$ for some $y \in G$. There exists $z \in G$ such that $nz = y$. Thus $x = f(y) = f(nz) = nf(z)$, and $f(z) \in H$, so that H is divisible.

2.2.2 THEOREM. Let $\{G_i\}_{i \in I}$ be a set of groups and let $G = \oplus \sum_{i \in I} G_i$. Then G is divisible if and only if G_i is divisible for every $i \in I$.

Proof: Suppose G is divisible. Let π_i denote the projection of G onto G_i. Then $G_i = \pi_i(G)$ is the homomorphic image of a divisible group, and so is divisible by 2.2.1.

Now suppose G_i is divisible for every $i \in I$. Let n be a nonzero integer and $x \in G$. Let π_i denote the projection of G onto G_i, and θ_i the injection of G_i into G. Then there exists a finite subset B of I such that $\pi_i(x) \neq 0$ if and only if $i \in B$, and $x = \sum_{i \in B} \theta_i(\pi_i(x))$.

Now for each $i \in B$, since G_i is divisible, there exists $y_i \in G_i$ such that $\pi_i(x) = ny_i$. Let $y = \sum_{i \in B} \theta_i(y_i)$. Then $ny = n(\sum_{i \in B} \theta_i(y_i)) = \sum_{i \in B} n\theta_i(y_i) = \sum_{i \in B} \theta_i(ny_i) = \sum_{i \in B} \theta_i(\pi_i(x)) = x$. Thus $nG = G$, and G is divisible.

Note that in general it is not true that subgroups of divisible groups are divisible, as can be seen from the example of the additive group of rational numbers which contains the integers as a nondivisible subgroup. However, as 2.2.2 shows, any subgroup of a divisible group which is also a direct summand of that group is divisible. Now we shall look at some subgroups which are direct summands under certain conditions.

2.2.3 DEFINITION. Let G be a group. The subgroup $G_t = \{x \in G \mid nx = 0 \text{ for some positive integer } n\}$ is called the torsion subgroup of G. If $G = G_t$, G is said to be a torsion group. If $G_t = 0$, G is said to be a torsion-free group.

It can be proved (8, p.193) that if G is a finitely generated abelian group, then G_t is a finite group, and G is isomorphic to $G_t \oplus G/G_t$. Thus there exists a torsion-free subgroup F of G, isomorphic to G/G_t, such that $G = G_t \oplus F$. A similar result can be proved for any divisible group, whether or not it is finitely generated. With this end in mind we prove the following two results.

2.2.4 PROPOSITION. Let G be a divisible group. Then G_t and G/G_t are both divisible groups.

Proof: Let n be a nonzero integer and $x \in G_t$. There exists $y \in G$ such that $ny = x$. But $mx = 0$ for some positive integer m, so that $|mn|y = 0$. Thus $y \in G_t$, and G_t is divisible. Finally, G/G_t is a homomorphic image of G and so is divisible by 2.2.1.

2.2.5 THEOREM. Let H be an injective group. Then H is a direct summand of every group which contains it.

Proof: Let G be a group which contains H. Let i denote the natural inclusion map from H into G, and let i' denote the identity map from H onto itself. Then since H is injective there exists a homomorphism f from G into H such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "i"] \arrow[d, "i'"'] & G \arrow[dl, "f"] \\ & H & \end{tikzcd}$$

commutes, i.e., such that $f(x) = x$ for every $x \in H$. We will show that $G = H \oplus \text{Ker } f$, where we define $\text{Ker } f = \{x \in G \mid f(x) = 0\}$. First of all, note that $H \cap \text{Ker } f = 0$. For suppose $x \in H \cap \text{Ker } f$, then $x = f(x) = 0$. Now let $a \in G$. Then $f(a) \in H$ and $f(f(a)) = f(a)$. Thus $a - f(a) \in \text{Ker } f$, since $f(a - f(a)) = f(a) - f(f(a)) = f(a) - f(a) = 0$. We have $a = f(a) + a - f(a) \in H + \text{Ker } f$. Thus $G = H \oplus \text{Ker } f$, and H is a direct summand of G.

The above theorem is the basis for much of the theory of divisible groups. A proof based on the divisibility of H, rather than the fact that it is injective, can be found in (7, pp.8-9).

The following result allows us in the next section to characterize all divisible abelian groups.

2.2.6 THEOREM. Let G be a divisible group with G_t its torsion subgroup. Then $G = G_t \oplus F$ where F is a torsion-free subgroup of G isomorphic to G/G_t.

Proof: By 2.2.4 G_t is divisible and thus a direct summand of G by 2.2.5. Thus there exists a subgroup F of G such that $G = G_t \oplus F$. Furthermore, F is clearly torsion-free since $G_t \cap F = 0$, and F is isomorphic to G/G_t by the Second Isomorphism Theorem.

Section 3: Characterization of Divisible Groups

2.3.1 THEOREM. Let F be a divisible torsion-free group. Then there exists an index set I such that $F = \oplus \sum_{i \in I} F_i$ where each F_i is isomorphic to Q, the additive group of rational numbers.

Proof: Let $x \in F$ and n be a nonzero integer. Since F is divisible there exists an element $y \in F$ such that $ny = x$. Furthermore this y is unique. For suppose $x = ny = ny'$ with $y, y' \in F$. Then $0 = ny - ny' = n(y - y')$ implying that $y - y' = 0$ since F is torsion-free. Thus $y = y'$. Consequently we can attach a unique meaning to $\frac{1}{n}x$ for each $x \in F$ and nonzero integer n, and hence also to rx where r is any rational number.

Let I' be a set of generators of F. If $x \in I'$ then $Qx = \{rx \mid r \in Q\} \subseteq F$ since $rx \in F$ for every $r \in Q$. Thus $\sum_{x \in I'} Qx \subseteq F$. Also $F = \sum_{x \in I'} Zx \subseteq \sum_{x \in I'} Qx$, since I' is a set of generators of F, and we have $F = \sum_{x \in I'} Zx = \sum_{x \in I'} Qx$. Let S be the set of all subsets I" of I' such that $\sum_{x \in I''} Qx$ is a direct sum. S is not empty since $\{x\} \in S$ for every $x \in I'$. Let the partial order on S be set inclusion. Let $\{I_j\}_{j \in J}$ be a chain in S. Let $I^* = \bigcup_{j \in J} I_j$. To verify that $I^* \in S$ we must show that $\sum_{x \in I^*} Qx$ is a direct sum. Thus we must show that

$Qy \cap \sum_{x \in I^* - \{y\}} Qx = 0$ for every $y \in I^*$. Let $y \in I^*$ and $z \in Qy \cap \sum_{x \in I^* - \{y\}} Qx$. Then $z = ry = \sum_{i=1}^{n} r_i x_i$ for some positive integer n, $r, r_i \in Q$, $x_i \in I^*$ $(x_i \neq y)$, for each $i = 1, \ldots, n$. Now because $\{y, x_1, \ldots, x_n\}$ is a finite set and I^* is the union of a chain of sets, there exists $j \in J$ such that $y \in I_j$ and $x_i \in I_j$ for every $i \in \{1, \ldots, n\}$. Thus $z \in Qy \cap \sum_{x \in I_j - \{y\}} Qx$ but $Qy \cap \sum_{x \in I_j - \{y\}} Qx = 0$ since $\sum_{x \in I_j} Qx$ is a direct sum. So $z = 0$, and we have $\sum_{x \in I^*} Qx$ a direct sum. Clearly I^* is an upper bound for the chain $\{I_j\}_{j \in J}$, so that we may apply Zorn's Lemma to get a maximal element, I, of S. If we can show that $F = \oplus \sum_{x \in I} Qx$ we will have proved the theorem, since each Qx is clearly isomorphic to Q. Suppose $F \neq \oplus \sum_{x \in I} Qx$. Let $y \in F$ such that $y \notin \oplus \sum_{x \in I} Qx$. We know that $Qy \cap \sum_{x \in I} Qx \neq 0$ by the maximality of I. Hence there exist integers n and m $(m \neq 0)$ such that $\frac{n}{m} y \in \oplus \sum_{x \in I} Qx$. It follows that $ny \in \oplus \sum_{x \in I} Qx$. Note that $\oplus \sum_{x \in I} Qx$ is divisible by Theorem 2.2.2, since each Qx is isomorphic to Q and therefore divisible. Thus there exists $z \in \oplus \sum_{x \in I} Qx$ such that $nz = ny$. Then $nz - ny = 0$, implying $n(z - y) = 0$. Since F is torsion-free, we have $z - y = 0$, or $y = z \in \oplus \sum_{x \in I} Qx$, a contradiction. Thus $F = \oplus \sum_{x \in I} Qx$ and the theorem is proved.

The following theorem gives us a characterization of the divisible torsion groups. The proof is based on Zorn's Lemma and may be found in (7, pp.10-11).

2.3.2 THEOREM. Let T be a divisible torsion group. Then there exists an index set I and a set of prime numbers $\{p_i\}_{i\in I}$ such that $T = \oplus \sum_{i\in I} Z(p_i^{\infty})$.

We have shown in Theorem 2.2.6 that any divisible group is a direct sum of a torsion group and a torsion-free group, so that by Theorems 2.3.1 and 2.3.2 any divisible group must be a direct sum of groups each isomorphic to the additive group of rational numbers or to $Z(p^{\infty})$ for various primes p. Furthermore, by Propositions 2.1.6 and 2.1.8 and Theorem 2.2.2 any group of this form is divisible. Thus we have a complete characterization of all divisible groups, summarized in the following theorem.

2.3.3 THEOREM. A group G is divisible if and only if there exist an index set I, with $\{p_i\}_{i\in I}$ a set of prime numbers, and an index set J such that

$$G = \oplus \sum_{i\in I} Z(p_i^{\infty}) \oplus \sum_{J} Q$$

where Q denotes the additive group of rational numbers.

Section 4: The Injective Envelope

2.4.1 DEFINITION. If G and H are abelian groups, with G a subgroup of H, then G is said to be essential in H provided if L is a nonzero subgroup of H then $G \cap L \neq 0$. If G is essential in H, H is said to be an essential extension of G. An essential extension H of G is called a maximal essential extension if given any other essential extension, K, of G then K can be embedded in H.

It is easily shown that any abelian group can be embedded in a divisible group (7, p.12). The following theorem, which is a corollary to a theorem due to Eckmann and Schopf, gives us a stronger result. The proof may be found in (3, pp.390-392).

2.4.2 THEOREM. (Corollary to Eckmann and Schopf Theorem) Let G be an abelian group. Then G may be embedded in an injective group which is simultaneously an essential extension of G, and which is called the injective envelope of G. The injective envelope of G is contained in any injective group which contains G. Furthermore, an abelian group H containing G is an injective envelope of G if and only if H is a maximal essential extension of G. If G and G' are abelian groups with injective envelopes H and H' respectively, and if ϕ is an isomorphism from G onto G', then ϕ can be extended to an isomorphism from H onto H'.

Note that it follows immediately from this theorem that an injective envelope of a group is unique up to isomorphism, so that we are justified in referring to "the" injective envelope of G, which we shall henceforth denote by E(G). The next four results on injective envelopes will be needed for our work in Chapter III.

<u>2.4.3 PROPOSITION</u>. Let $\{G_i\}_{i\in I}$ be a sequence of groups and let $E(G_i)$ denote the injective envelope of G_i for each $i \in I$. Let $G = \oplus \sum_I G_i$. Then the injective envelope of G is given by $E(G) = \oplus \sum_I E(G_i)$.

Proof: By Theorem 2.2.2, $\oplus \sum_I E(G_i)$ is divisible and therefore injective. Thus $E(G) \subseteq \oplus \sum_I E(G_i)$. Now $G_i \subseteq \oplus \sum_I G_i \subseteq E(G)$ so that $E(G_i) \subseteq E(G)$ for every $i \in I$. Thus $\oplus \sum_I E(G_i) \subseteq E(G)$, and we have $E(G) = \oplus \sum_I E(G_i)$.

<u>2.4.4 PROPOSITION</u>. (a) Any essential extension of a torsion-free group is a torsion-free group. (b) Any essential extension of a torsion group is a torsion group.

Proof: (a) Let G be a torsion-free group and H an essential extension of G. Suppose the torsion subgroup of H, H_t, is not 0. Then $G \cap H_t \neq 0$. Choose nonzero $x \in G \cap H_t$. Then there exists a positive integer n such that $nx = 0$, contradicting the fact that G is torsion-free. Thus $H_t = 0$ and H is also torsion-free.

(b) Let G be a torsion group and H an essential extension of G. Suppose $H \neq H_t$. Choose nonzero $h \in H$ such that $h \notin H_t$. Let H' be the subgroup of H generated by h. Then $H' \cap G \neq 0$. Choose nonzero $g \in H' \cap G$. Now $g = mh$ for some positive integer m since $g \in H'$. Also there exists a positive integer n such that $ng = 0$ since $g \in G$, a torsion group. Thus $ng = nmh = 0$, contradicting the fact that $h \notin H_t$. Thus $H = H_t$ and H is a torsion group.

Note that it follows from this result that the injective envelope of a torsion-free group is torsion-free, and the injective envelope of a torsion group is a torsion group.

We now wish to show that the injective envelope of $Z(p^n)$ is $Z(p^\infty)$ for any prime p and positive integer n. To do this we need the following lemma.

2.4.5 LEMMA. Let p be a prime number and let G be a nonzero proper subgroup of $Z(p^\infty)$. Then $G = Z(p^k)$ for some positive integer k.

Proof: First note that any element of the form $\frac{m}{p^j} + Z$, where m is an integer relatively prime to p^j, will generate the group $Z(p^j)$. To see this, let $\frac{x}{p^j} + Z \in Z(p^j)$. Since m and p^j are relatively prime, there exist integers a and b such that $am + bp^j = 1$. Thus

$$\frac{x}{p^j} + Z = \frac{(am + bp^j)x}{p^j} + Z = \frac{amx}{p^j} + \frac{bp^jx}{p^j} + Z =$$

$$\frac{amx}{p^j} + Z = \frac{(ax)m}{p^j} + Z = ax(\frac{m}{p^j} + Z),$$

so that $\frac{m}{p^j} + Z$ does indeed generate $Z(p^j)$.

Now let k be the smallest positive integer such that $\frac{1}{p^k} + Z \notin G$. Such an integer exists, otherwise $G = Z(p^\infty)$. Note that $\frac{1}{p^{k+i}} + Z \notin G$ for any positive integer i, since otherwise $p^i(\frac{1}{p^{k+i}} + Z) = \frac{1}{p^k} + Z \in G$. We claim that $G = Z(p^{k-1})$. For suppose $X \in Z(p^\infty)$ such that $X \notin Z(p^{k-1})$. Then we can write $X = \frac{m}{p^{k+i}} + Z$ for some non-negative integer i and integer m relatively prime to p^k. Thus X generates the group $Z(p^{k+i})$. Since $\frac{1}{p^{k+i}} + Z \in Z(p^{k+i})$ and $\frac{1}{p^{k+i}} + Z \notin G$, we see that $X \notin G$. Thus $G \subseteq Z(p^{k-1})$.

But we already know that $\frac{1}{p^{k-1}} + Z \in G$ by choice of k. Thus $Z(p^{k-1}) \subseteq G$, since $\frac{1}{p^{k-1}} + Z$ generates $Z(p^{k-1})$. So we see that $G = Z(p^{k-1})$ and the Lemma is proved.

2.4.6 PROPOSITION. Let p be a prime number and n a positive integer; then the injective envelope of $Z(p^n)$ is $Z(p^\infty)$.

Proof: Let $E(Z(p^n))$ denote the injective envelope of $Z(p^n)$. By Proposition 2.1.8, $Z(p^\infty)$ is divisible, and thus injective, so that $E(Z(p^n)) \subseteq Z(p^\infty)$. Since $E(Z(p^n))$ is a nonzero subgroup of $Z(p^\infty)$ we must have $E(Z(p^n)) = Z(p^k)$ for some positive integer k or $E(Z(p^n)) = Z(p^\infty)$, by Lemma 2.4.5. But we cannot have $E(Z(p^n)) = Z(p^k)$, for by Theorem 2.3.3 we know that $Z(p^k)$ is not divisible for any positive integer k. Thus we must have $E(Z(p^n)) = Z(p^\infty)$, and the proposition is proved.

CHAPTER III

QUASI-INJECTIVE ABELIAN GROUPS

Section 1: Definition and Important Theorems

3.1.1 DEFINITION. An abelian group G is said to be quasi-injective provided if $0 \longrightarrow L \xrightarrow{f} G$ is an exact sequence of abelian groups and if g is a group homomorphism from L into G then there exists a group homomorphism h from G into itself such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & G \arrow[dl, "h"] \\ & G & \end{tikzcd}$$

commutes. Again we get an equivalent definition if we assume L is a subgroup of G and f is the natural injection of L into G.

Note that it follows immediately from the definitions that any injective group is also quasi-injective. The following theorem is due to Johnson and Wong (6) and will be used frequently in later proofs.

3.1.2 THEOREM. Let G be an abelian group and $E(G)$ its injective envelope. Then G is quasi-injective if and only if $\lambda(G) \subseteq G$ for every homomorphism λ from $E(G)$ into itself.

Proof: Suppose that $\lambda(G) \subseteq G$ for every homomorphism λ of $E(G)$ into itself. Let L be a subgroup of G and f a homomorphism from L into G. Let i denote the inclusion map from L into G, and i' the inclusion map from G into $E(G)$. Then because $E(G)$ is injective there exists a homomorphism g from $E(G)$ into itself such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "f"'] & G \arrow[r, "i'"] & E(G) \arrow[ddll, "g"] \\
& G \arrow[d, "i'"'] & & \\
& E(G) & &
\end{tikzcd}$$

commutes. Thus $g(x) = f(x)$ for every $x \in L$. Now $g(G) \subseteq G$ by hypothesis, so that the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "f"'] & G \arrow[dl, "g'"] \\
& G &
\end{tikzcd}$$

where g' is the function g restricted to G, also commutes. Therefore G is quasi-injective.

Conversely, suppose that G is quasi-injective. Let λ be a homomorphism from E(G) into itself. Let $N = \{x \in G \mid \lambda(x) \in G\}$. Clearly N is a subgroup of G. If we let λ' denote the function λ restricted to N, i the natural inclusion map from N into G, and i' the natural inclusion map from G into E(G), we see that, since G is quasi-injective, there exists a homomorphism φ from G into itself such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\lambda'"'] & G \arrow[r, "i'"] \arrow[dl, "\varphi"] & E(G) \arrow[ddll, "\lambda"] \\
& G \arrow[d, "i'"'] & & \\
& E(G) & &
\end{tikzcd}$$

commutes. Thus we have $\lambda'(n) = \lambda(n) = \varphi(n)$ for every $n \in N$.

We want to show that N = G. To show this it suffices to show that $(\lambda-\varphi)(G) = 0$. We have already seen that $(\lambda-\varphi)(N) = 0$. Suppose $(\lambda-\varphi)(G) \neq 0$. Since $(\lambda-\varphi)(G)$ is a subgroup of E(G) and E(G) is an essential extension of G, we see that $(\lambda-\varphi)(G) \cap G \neq 0$. Choose nonzero $y \in (\lambda-\varphi)(G) \cap G$. Then there exists $x \in G$ such that $(\lambda-\varphi)(x) = y \in G$. Thus $\lambda(x) = \varphi(x) + y \in G$, implying $x \in N$, so that $y = (\lambda-\varphi)(x) = 0$, a contradiction. Therefore $(\lambda-\varphi)(G) = 0$, implying $\lambda(G) \subseteq G$, and the theorem is proved.

<u>3.1.3 THEOREM</u>. Let G be a quasi-injective group such that $G = \oplus \sum_{i \in I} G_i$; then G_i is quasi-injective for every $i \in I$.

Proof: Let E(G) denote the injective envelope of G and $E(G_i)$ the injective envelope of G_i for each $i \in I$. By Proposition 2.4.3, $E(G) = \oplus \sum_{i \in I} E(G_i)$. Choose $i \in I$ and λ_i a homomorphism from $E(G_i)$ into itself. Let π_i denote the natural projection of E(G) onto $E(G_i)$ and θ_i the natural injection of $E(G_i)$ into E(G). Define a mapping λ from E(G) into itself by $\lambda(x) = \theta_i \lambda_i \pi_i(x)$ for every $x \in E(G)$. Since λ is a composition of homomorphisms, it is a homomorphism. Because G is quasi-injective $\lambda(G) \subseteq G$ by Theorem 3.1.2. Thus we have $\lambda(G) = \theta_i \lambda_i \pi_i(G) = \theta_i \lambda_i (G_i) \subseteq G = \oplus \sum_{i \in I} G_i$, so that $\pi_i \theta_i \lambda_i (G_i) \subseteq \pi_i (\oplus \sum_{i \in I} G_i)$. That is, $\lambda_i(G_i) \subseteq G_i$ and G_i is quasi-injective.

We remark that, unlike the case for injective groups, the converse of Theorem 3.1.3 is false. That is, if G_i is quasi-injective for every $i \in I$ it does not follow that $G = \oplus \sum_{i \in I} G_i$ is quasi-injective. We will see examples of this later.

<u>3.1.4 THEOREM</u>. Let G be a quasi-injective group. Then $G = G_t \oplus F$ where G_t is the torsion subgroup of G and F is a torsion-free group.

Proof: Let $E(G)$ be the injective envelope of G. By Theorem 2.2.6 we can write $E(G) = T' \oplus F'$ where T' is the torsion subgroup of $E(G)$ and F' is a torsion-free subgroup of $E(G)$. Let $\pi_{T'}$ be the projection map from $E(G)$ onto T'. Since $\pi_{T'}$ is a homomorphism from $E(G)$ into itself and since G is quasi-injective, we know by Theorem 3.1.2 that $\pi_{T'}(G) \subseteq G$. Similarly letting $\pi_{F'}$ be the projection map from $E(G)$ onto F' we get the result $\pi_{F'}(G) \subseteq G$. Now let $g \in G$ so that $g = t + f$ with $t \in T'$ and $f \in F'$. We know $\pi_{T'}(g) = t \in G \cap T'$ and $\pi_{F'}(g) = f \in G \cap F'$ so that $G = (G \cap T') + (G \cap F')$. But $(G \cap T') \cap (G \cap F') = 0$ since $T' \cap F' = 0$, so that $G = (G \cap T') \oplus (G \cap F')$. Clearly $G \cap F'$ is a torsion-free group and $G \cap T' = G_t$.

Section 2: Examples of Quasi-injective Groups and Groups that are not Quasi-injective

3.2.1 PROPOSITION. Let p be a prime number and n a positive integer; then $Z(p^n)$ is quasi-injective.

Proof: By Proposition 2.4.6 the injective envelope of $Z(p^n)$ is $Z(p^\infty)$. Let λ be a homomorphism from $Z(p^\infty)$ into itself. Note that $\lambda(Z(p^n))$ is a proper subgroup of $Z(p^\infty)$. Thus if we denote the zero group by $Z(p^0)$, we have by Lemma 2.4.5 that $\lambda(Z(p^n)) = Z(p^k)$ for some non-negative integer k. Since the order of $\lambda(Z(p^n))$ is less than or equal to the order of $Z(p^n)$, we must have $k \leq n$. Thus $\lambda(Z(p^n)) \subseteq Z(p^n)$ and $Z(p^n)$ is quasi-injective by Theorem 3.1.2.

3.2.2 PROPOSITION. Let p be a fixed prime number, n a fixed positive integer, A an index set, and let $G = \oplus \sum_A Z(p^n)$; then G is quasi-injective.

Proof: By Propositions 2.4.3 and 2.4.6 the injective envelope of G is given by $E(G) = \oplus \sum_A Z(p^\infty)$. Let λ be a homomorphism from E(G) into itself. Then $\lambda(G)$ is a subgroup of E(G). Let $a \in A$ and let π_a denote the natural projection map from E(G) onto its a^{th} coordinate. Then

$\pi_a(\lambda(G))$ is a subgroup of $Z(p^\infty)$, so that $\pi_a(\lambda(G)) = Z(p^{m_a})$ for some non-negative integer m_a or $m_a = \infty$, by Lemma 2.4.5. Now every element in G has order a divisor of p^n. It follows that every element in $\lambda(G)$ and thus in $\pi_a(\lambda(G))$ has order a divisor of p^n. Thus $m_a \leq n$. Finding similar m_a for each $a \in A$ we have the result $\lambda(G) \subseteq \oplus \sum_{a\in A} \pi_a(\lambda(G)) = \oplus \sum_{a\in A} Z(p^{m_a}) \subseteq \oplus \sum_{A} Z(p^n) = G$. Thus G is quasi-injective by Theorem 3.1.2.

<u>3.2.3 PROPOSITION</u>. Let I be an index set such that:

(1) $\{p_i\}_{i\in I}$ is a set of distinct primes,

(2) n_i is a positive integer or $n_i = \infty$ for every $i \in I$,

(3) A_i is a set for every $i \in I$,

(4) $G_i = \oplus \sum_{A_i} Z(p_i^{n_i})$;

then $G = \oplus \sum_{i\in I} G_i$ is quasi-injective.

Proof: By Propositions 2.4.3 and 2.4.6 the injective envelope of G_i is given by $E(G_i) = \oplus \sum_{A_i} Z(p_i^\infty)$ for every $i \in I$, and the injective envelope of G is given by $E(G) = \oplus \sum_{i\in I} E(G_i)$. Let π_i denote the natural projection of $E(G)$ onto $E(G_i)$, and θ_i the natural injection of $E(G_i)$ into $E(G)$. Let λ be a nonzero homomorphism from $E(G)$ into itself. Note that $\theta_i\pi_i(E(G)) = \theta_i(E(G_i))$ is a subgroup of $E(G)$ isomorphic to $E(G_i)$, so that we can identify

$\theta_i\pi_i(G)$ with G_i. First we want to show that λ restricted to $E(G_i)$ is a homomorphism from $E(G_i)$ into itself. To see this let $x \in E(G_i) = \oplus \sum_{A_i} Z(p_i^{\infty})$. Then x has order a power of p_i, say $p_i^{m_i}$. We want to show that $\lambda(x) \in E(G_i)$. This is true if and only if $\pi_j\lambda(x) = 0$ for every $j \in I$ such that $j \neq i$. Since $\pi_j\lambda(x) \in \pi_j(E(G)) = E(G_j) = \oplus \sum_{A_j} Z(p_j^{\infty})$, then $\pi_j\lambda(x)$ has order a power of p_j. But $p_i^{m_i}\pi_j\lambda(x) = \pi_j\lambda(p_i^{m_i}x) = \pi_j\lambda(0) = 0$, so that the order of $\pi_j\lambda(x)$ divides $p_i^{m_i}$ also. Since $p_j \neq p_i$, we must have the order of $\pi_j\lambda(x)$ equal to one, i.e., $\pi_j\lambda(x) = 0$ for $i \neq j$. Thus $\lambda(x) \in E(G_i)$, and so the restriction of λ to $E(G_i)$ is a homomorphism from $E(G_i)$ into itself for every $i \in I$. Thus since G_i is quasi-injective by Proposition 3.2.2, we have $\lambda(G_i) \subseteq G_i$ for every $i \in I$. Now let $g \in G = \oplus \sum_{i \in I} G_i$. Then $g = \sum_{i \in I} \theta_i\pi_i(g)$, implying that $\lambda(g) = \lambda(\sum_{i \in I} \theta_i\pi_i(g)) = \sum_{i \in I} \lambda\theta_i\pi_i(g)$. Since $\lambda\theta_i\pi_i(g) \in \lambda(G_i) \subseteq G_i$, we have $\lambda(g) \in \oplus \sum_{i \in I} G_i = G$, i.e., $\lambda(G) \subseteq G$ and G is quasi-injective.

The following two propositions give examples, promised earlier, of cases when the direct sum of quasi-injective groups is not quasi-injective.

<u>3.2.4 PROPOSITION</u>. Let p be a prime number, m and n positive integers such that $m \neq n$, and let $G = Z(p^n) \oplus Z(p^m)$. Then G is not quasi-injective.

Proof: By Propositions 2.4.3 and 2.4.6 the injective envelope of G is given by $E(G) = Z(p^\infty) \oplus Z(p^\infty)$. Define a mapping λ from E(G) into itself as follows. Let $g = (g_1,g_2) \in Z(p^\infty) \oplus Z(p^\infty)$. Define $\lambda(g_1,g_2) = (g_2,g_1)$. It is easily seen that λ is a homomorphism. However $\lambda(G) = Z(p^m) \oplus Z(p^n) \nsubseteq Z(p^n) \oplus Z(p^m) = G$ since $m \neq n$. Thus G is not quasi-injective.

<u>3.2.5 PROPOSITION</u>. Let p be a prime number and n a positive integer, and let Q denote the additive group of rational numbers; then $G = Z(p^n) \oplus Q$ is not quasi-injective.

Proof: The injective envelope of G is given by $E(G) = Z(p^\infty) \oplus Q$. We will construct a homomorphism from E(G) into itself such that $\lambda(G) \nsubseteq G$. Choose an integer $k > n$ and look at the diagram

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "i"] \arrow[d, "f"'] & Q \\ & Z(p^k) \arrow[d, "i'"'] & \\ & Z(p^\infty) & \end{tikzcd}$$

where i denotes the natural injection of the additive group of integers, Z, into Q, i' denotes the natural injection of $Z(p^k)$ into $Z(p^\infty)$, and f denotes the homomorphism from Z onto $Z(p^k)$ defined by $f(m) = \frac{m}{p^k} + Z$ for every integer m. Since $Z(p^\infty)$ is injective, there exists a homomorphism h from Q into $Z(p^\infty)$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "i"] \arrow[d, "f"] & Q \arrow[ddl, "h"] \\
& Z(p^k) \arrow[d, "i'"] & \\
& Z(p^\infty) &
\end{tikzcd}$$

commutes. Note that $Z(p^k) = i'(Z(p^k)) = i'f(Z) = hi(Z) = h(Z) \subseteq h(Q)$. Define a mapping λ from $E(G) = Z(p^\infty) \oplus Q$ into itself by $\lambda(x,y) = (h(y),y)$ for every $x \in Z(p^\infty)$ and $y \in Q$. It is easy to see that λ is a homomorphism. We have $\lambda(G) = \lambda(Z(p^n) \oplus Q) = h(Q) \oplus Q$. But $h(Q) \not\subseteq Z(p^n)$ since $Z(p^n) \subsetneq Z(p^k) \subseteq h(Q)$, so that $\lambda(G) = h(Q) \oplus Q \not\subseteq Z(p^n) \oplus Q = G$. Thus by Theorem 3.1.2 G is not quasi-injective.

Section 3: Characterization of Quasi-injective Groups

3.3.1 PROPOSITION. Let G be a quasi-injective torsion-free abelian group. Then G is injective.

Proof: Let $E(G)$ be the injective envelope of G. By Proposition 2.4.4 (a), $E(G)$ is also torsion-free. Since $E(G)$ is both divisible and torsion-free we can uniquely define $\frac{1}{n}g$ for every $g \in E(G)$ and nonzero integer n, as was shown in the proof of Theorem 2.3.1. Let n be any nonzero integer and define a mapping λ_n from $E(G)$ into itself by $\lambda_n(g) = \frac{1}{n}g$ for every $g \in E(G)$. It is easily seen that λ_n is a homomorphism. Thus $\lambda_n(G) \subseteq G$ by Theorem 3.1.2 since G is quasi-injective. Let $g \in G$; then $\lambda_n(g) = \frac{1}{n}g \in G$, and $n(\frac{1}{n}g) = g$, so that $nG = G$ for every nonzero integer n. Thus G is divisible and so injective.

3.3.2 THEOREM. A group G is a quasi-injective torsion group if and only if there exists an index set I, and sets A_i, $i \in I$, such that:

(1) $\{p_i\}_{i\in I}$ is a set of distinct primes,

(2) n_i is a positive integer or $n_i = \infty$ for each $i \in I$,

(3) $G \cong \oplus \sum_{i\in I} G_i$ where $G_i = \oplus \sum_{A_i} Z(p_i^{\,n_i})$.

Proof: Assume that G is a quasi-injective torsion group. Since G is a torsion group its injective envelope $E(G)$ is also a torsion group by Proposition 2.4.4 (b). Thus by Theorem 2.3.2 there exists an index set I' such that $\{p_i\}_{i\in I'}$ is a set of prime numbers and $E(G) = \oplus \sum_{i\in I'} Z(p_i^{\infty})$. Let π_i denote the natural projection of $E(G)$ onto $Z(p_i^{\infty})$. It is easily seen that $G \subseteq \oplus \sum_{i\in I'} \pi_i(G)$. If we identify $Z(p_i^{\infty})$ with its isomorphic image in $E(G)$, then π_i is a homomorphism of $E(G)$ into itself, so that by Theorem 3.1.2, $\pi_i(G) \subseteq G$ for every $i \in I'$. Thus $\oplus \sum_{i\in I'} \pi_i(G) \subseteq G$, and we have $G = \oplus \sum_{i\in I'} \pi_i(G)$. Now $\pi_i(G)$ is a subgroup of $\pi_i(E(G)) = Z(p_i^{\infty})$, and so by Lemma 2.4.5 we have $\pi_i(G) = Z(p_i^{n_i})$ for n_i some positive integer or $n_i = \infty$. Thus $G = \oplus \sum_{i\in I'} Z(p_i^{n_i})$ where each n_i is a positive integer or ∞. However $\{p_i\}_{i\in I'}$ may not be a set of distinct primes. But note that if $p_i = p_j$ then $n_i = n_j$. For we can write $G \cong G' \oplus Z(p_i^{n_i}) \oplus Z(p_j^{n_j})$ for some group G', and since G is quasi-injective we must have $Z(p_i^{n_i}) \oplus Z(p_j^{n_j})$ quasi-injective by Theorem 3.1.3. Thus $n_i = n_j$ by Proposition 3.2.4. Now define $A_i = \{j \in I' \mid p_j = p_i\}$ for every $i \in I'$, and choose $I \subseteq I'$ such that $\bigcup_{i\in I} A_i = I'$ and $A_i \cap A_j = \emptyset$ if $i \neq j$ for $i,j \in I$. Clearly we have $\{p_i\}_{i\in I}$ a set of distinct primes, and n_i a positive integer or ∞ for every $i \in I$. Letting $G_i = \oplus \sum_{A_i} Z(p_i^{n_i})$ we have $G \cong \oplus \sum_{i\in I} G_i$.

Since the converse follows from Proposition 3.2.3, the Theorem is proved.

3.3.3 THEOREM. Let G be a quasi-injective group which is not injective; then G is a torsion group.

Proof: Since G is quasi-injective, by Theorem 3.1.4 we can write $G = G_t \oplus F$ where G_t is the torsion subgroup of G and F is a torsion-free group. Suppose $F \neq 0$, then by Theorem 3.1.3 F is quasi-injective and thus injective by Proposition 3.3.1. Also by Theorem 3.1.3 G_t is quasi-injective. However, G_t cannot be injective, for then G would be injective by Theorem 2.2.2, contrary to hypothesis. Thus by Theorem 3.3.2 G is isomorphic to a group containing $Z(p^k)$ as a direct summand for some prime p and positive integer k. Since F is injective it is a direct sum of groups isomorphic to Q, the additive group of rational numbers. So we can write $G \cong G' \oplus Z(p^k) \oplus Q$ for some group G', and $Z(p^k) \oplus Q$ is quasi-injective by Theorem 3.1.3, contradicting Proposition 3.2.5. Thus our assumption that $F \neq 0$ is not valid, and we have G a torsion group.

SUMMARY

By Theorem 3.3.3 any quasi-injective group is either injective, in which case it is characterized in Theorem 2.3.3, or else it is a torsion group and is characterized in Theorem 3.3.2. These results are summarized in the following theorem.

<u>THEOREM</u>. Let G be a quasi-injective abelian group. Then either

(1) G is injective, in which case there exists an index set I, with $\{p_i\}_{i\in I}$ a set of prime numbers, and an index set J such that

$$G \cong \oplus \sum_{i\in I} Z(p_i^{\infty}) \oplus \sum_J Q$$

where Q denotes the additive group or rational numbers, or

(2) G is a torsion group and there exists an index set I, and sets A_i, $i\in I$, such that

(a) $\{p_i\}_{i\in I}$ is a set of distinct prime numbers,

(b) n_i is a positive integer or $n_i = \infty$ for each $i \in I$,

and

(c) $G \cong \oplus \sum_{i\in I} G_i$ where for each $i \in I$, $G_i = \oplus \sum_{A_i} Z(p_i^{n_i})$.

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